Direct Products and Sums

Q : Is it possible decompose a group into simpler groups?

Example $(\mathbb{R}^2, +)$ is constructed using two copies of $(\mathbb{R}, +)$.

composition in $\mathbb{R}^2$ composition in $\mathbb{R}$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}, \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Definition Let G and H be groups. The <u>direct product</u> of G and H is the set $G \times H$, together with the composition

composition in G composition in H

$$(g_1, h_1) * (g_2, h_2) := (g_1 g_2, h_1 h_2).$$

Tedious Exercise : $G \times H$ is a group. $e_{G \times H} = (e_G, e_H)$

$(g, h)^{-1} = (g^{-1}, h^{-1})$

Remarks

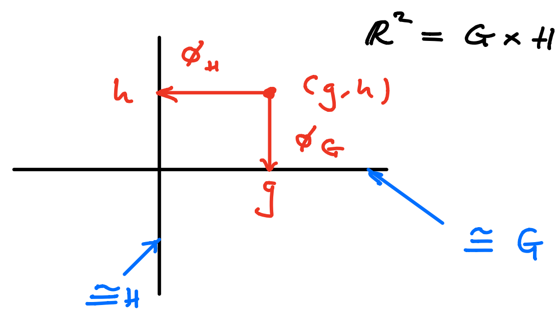
- $G \times H$ comes with natural homomorphisms

$$\begin{tikzcd} & G \times H \arrow[dl, "\phi_G"'] \arrow[dr, "\phi_H"] & \\ G & & H \end{tikzcd} \qquad \begin{tikzcd} & (g, h) \arrow[dl] \arrow[dr] & \\ g & & h \end{tikzcd}$$

Called a projection homomorphism

$$\left.\begin{aligned} \mathrm{Ker}(\phi_G) &= \{(e_G, h) \mid h \in H\} \cong H \\ \mathrm{Ker}(\phi_H) &= \{(g, e_H) \mid g \in G\} \cong G \end{aligned}\right\} \Rightarrow$$ G, H naturally isomorphic to subgroups of $G \times H$

Example $G = H = (\mathbb{R}, +)$



• All this can be generalized to any collection of groups.

In the finite case,

Same term by term composition.

$G_1, \dots, G_n$ groups $\Rightarrow$ $G_1 \times \dots \times G_n$ a group.

Subgroup of $G_1 \times \dots \times G_n$

Again $G_i \cong \{(e_{G_1}, \dots, e_{G_{i-1}}, g_i, e_{G_{i+1}}, \dots, e_{G_n}) \mid g_i \in G_i\}$

For example $G_i = (\mathbb{R}, +) \Rightarrow G_1 \times \dots \times G_n = (\mathbb{R}^n, +)$

• $G_1 \times G_2 \times \dots \times G_n$ Abelian $\Leftrightarrow$ G_i Abelian $\forall i \in \{1, \dots, n\}$

Q: When is a group isomorphic to some direct product?

<u>Definition</u> Let G be a group and $H_1, H_2, \dots, H_n \subset G$ be subgroups. We say G is a <u>direct sum</u> of $H_1, \dots, H_n$ if

1) Given $g \in G$, $\exists!$ $h_i \in H_i$ $\forall i \in \{1, \dots, n\}$ such that

There exist unique

$$g = h_1 * h_2 * h_3 * \dots * h_n$$

The H_i are perhaps non-Abelian

2) $h_i \in H_i$, $h_j \in H_j$, $i \neq j \Rightarrow h_i * h_j = h_j * h_i$

In this case we write $G = H_1 \oplus H_2 \oplus \dots \oplus H_n$

<u>Example</u>

V - real vector space, $\{\underline{v}_1, \dots, \underline{v}_n\} \subset V$, $\underline{v}_i \neq \underline{0}$

$G = (V, +)$, $H_i = (\mathrm{Span}(\underline{v}_i), +)$, then

$G = H_1 \oplus \dots \oplus H_n \Leftrightarrow \{\underline{v}_1, \dots, \underline{v}_n\}$ a <u>basis</u> for V

Proposition If 2/ is true, then 1/ is equivalent to the following:

1'/ Given $g \in G$, $\exists\, h_i \in H_i$ $\forall i \in \{1, \dots, n\}$

$$g = h_1 * h_2 * h_3 * \dots * h_n$$

Similar to spanning in a vector space

and

Similar to linear independence

$$h_1 * h_2 * \dots * h_n = e \implies h_1 = h_2 = \dots = h_n = e$$

Proof

Assume 2/ and 1/ are true. Let $h_i \in H_i$ and $h_1 * \dots * h_n = e$

uniqueness

$$\underbrace{e * e * e \dots * e}_{n \text{ times}} = e \implies h_1 = h_2 = \dots = h_n = e \implies 1'/ \text{ true}$$

Assume 2/ and 1'/ are true. Let $g_i, h_i \in H_i$ such that

$$g_1 * g_2 * \dots * g_n = h_1 * h_2 * \dots * h_n$$

$$\implies (g_1 * g_2 * \dots * g_n)^{-1} * (h_1 * \dots * h_n) = e$$

$$\implies (g_1^{-1} * h_1) * (g_2^{-1} * h_2) * \dots * (g_n^{-1} * h_n) = e \quad \text{and } g_i^{-1} * h_i \in H_i$$

$$\implies g_i^{-1} * h_i = e \implies g_i = h_i$$

$\implies$ 1/ true $\square$

Proposition

$$G = H_1 \oplus \dots \oplus H_n \implies G \cong H_1 \times H_2 \times \dots \times H_n$$

Proof

Define $\phi : H_1 \times H_2 \times \dots \times H_n \longrightarrow G$

$$(h_1, h_2, \dots, h_n) \longmapsto h_1 * h_2 * h_3 \dots * h_n$$

2/ $\implies \phi$ a homomorphism

1/ $\implies \phi$ a bijection $\square$